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The Module Structure of a Group Action on a Polynomial Ring

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For any representation of a p-group G on a vector space of dimension 3 over a finite field k of characteristic p, we show how the symmetric algebra, regarded as a kG-module, can be expressed as a direct sum of kG-modules, each one of which is isomorphic to a summand in low degree. It follows that, for any group G, only a finite number of isomorphism classes of summands can occur. © 1999 Academic Press

1. INTRODUCTION

1.1. *Results*

Let k be a finite field of $q = p^l$ elements, Let G be a p-group, and let M be a kG-module of dimension 3. We denote by S the symmetric algebra on M. This is, of course, equivalent to letting S be the polynomial ring in three variables, $S = k[x, y, z]$, and stating that G acts by graded ring automorphisms over k, and that the action on the homogeneous part of degree 1 is isomorphic to M. We are concerned with describing S as explicitly as possible as a kG-module.

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The invariant ring S^G is known to contain a polynomial ring $k[d_x, d_y, d_z]$, where d_x, d_y, d_z have degrees $1, q, q^2$, respectively. Our main structure theorem is the following:

THEOREM (Theorem 4.1). *There is an isomorphism of kG-modules*

$$S \approx (k[d_z] \otimes_k B) \oplus (k[d_x, d_z] \otimes_k L) \oplus \left(k\left[d_y, d_z\right] \otimes_k R\right) \\ \oplus \left(k\left[d_y, d_z\right] \otimes_k P\right).$$

Here $B = \bigoplus_{r=0}^{q^2-3} S^r$, $L = L^{q^2-2}$, $R = \bigoplus_{r=q^2-2}^{q^2+q-3} R^r$, *and* $P \approx kG$, *for certain kG-submodules* $L^{q^2-2} \subset S^{q^2-2}$, $R^r \subset S^r$, *and* $P \subset S^{q^2+q-2}$.

While this theorem looks somewhat technical as stated, we note that it is actually quite simple to interpret: it says that there are four types of indecomposable kG-modules in S. The first type of module is "propagated" only by the invariant d_z, the second type of module is propagated by the invariants d_z and d_x, the third type of module is propagated by the invariants d_z and d_y, and the fourth type is propagated by all three generators of the invariant ring.

As a surprising consequence of this result we obtain:

COROLLARY (Corollaries 4.2 and 4.3). *For any group H (not necessarily a p-group) and any kH-module M with* $\dim_k M \leq 3$, *the symmetric algebra of M contains only a finite number of isomorphism types of indecomposable summands as a kH-module.*

Furthermore, if H is a p-group, then every isomorphism type occurring appears in degree less than or equal to $q^2 + q - 2$. *In fact each* S^r *is a direct sum of a free module and certain* S^j *for* $j \leq q^2 + q - 3$, *and the multiplicities of the* S^j *are easily calculated.*

This shows that the a priori infinite problem of describing all the kG-module summands of the symmetric algebra reduces to the finite problem of describing all the summands up to a certain degree.

1.2. *Background*

Such decompositions of a symmetric algebra have been useful in a variety of contexts. They are of great usefulness in certain types of computations that arise in the cohomology of groups and its applications in homotopy theory. For example, the papers [11, 16, 22] use this technique. There are also connections between such decompositions and invariant theory, and we will make use of certain facts from invariant theory to prove the main theorem of this paper.

One of the first decomposition theorems of the type we discuss was proved by Almkvist and Fossum [1]. They gave an inductive description of the symmetric powers of any finite-dimensional $\mathbb{F}_p C$-module, for C a finite cyclic p-group. Shortly afterwards, a fascinating theorem was proved by D. Glover. We state here a special case that gives the essence of his work from our point of view:

THEOREM. *Let k be the prime field $\mathbb{F}_p$, let $G = SL_2(k)$, let $M \approx k^2$ be the natural kG-module, and let S be the symmetric algebra of M. Let T^r be S^r modulo a largest projective summand. Then the sequence $\{T^r\}_{r \in \mathbb{N}}$ is periodic with period $p(p-1)$, and each module T^r is indecomposable.*

Glover's result was in fact somewhat more detailed than this and applied also to $G = GL_2(k)$. Even the stripped-down version we have given here, however, describes implicitly a G-decomposition of S. The interested reader may consult the original paper [15].

Alperin and Kovacs [3] subsequently showed how to generalize this result to one for $k = \mathbb{F}_q$, i.e., the case where k is *not* necessarily the prime field, and found that the sequence $\{T^r\}_{r \in \mathbb{N}}$ is $q(q-1)$-periodic for these k. Their result, like Glover's, gives an essentially complete description of the G-decomposition of S.

These results, and other similar decompositions, provoke the natural question of why such decompositions have the favorable properties that they so often do. In particular, Siegel observed that in all the examples of such decompositions that had been computed in the setting of this paper, only a finite number of isomorphism types of indecomposable modules appeared in the G-decomposition of S, and asked whether this was always the case. For the reader wondering why this is not a triviality, it is worth pointing out that for a general finite group G, the indecomposable kG-modules have not been classified. In fact, even for $k[\mathbb{Z}/2 \times \mathbb{Z}/2]$ in characteristic 2, there are infinitely many indecomposable modules and the classification is quite involved [17, 4].

As this type of finiteness result is one of the aims of our work here, we formalize it into the following conjecture:

Conjecture (*Siegel's Finiteness Conjecture*). In the setting of this paper (i.e., k, G, M, S as above, except M may be of any dimension), a G-decomposition of S requires only a finite number of isomorphism types of indecomposable modules.

While Siegel has never published this conjecture and is careful always to refer to it as a question, we feel that his work on certain special cases of the conjecture justifies our nomenclature. For example, Siegel and Totaro [23] gave decompositions for every $k[\mathbb{Z}/2 \times \mathbb{Z}/2]$-module in characteristic 2, by studying the symmetric algebras of the (already classified) indecomposable modules and using the elementary fact that $S^*(M \oplus N) \approx S^*(M) \otimes S^*(N)$.

The reader may have already wondered what can be said about extending the results of Glover and Alperin and Kovacs to $SL_n(k)$. Indeed, this is the direction we are pursuing here. It is worth remarking that in the case k algebraically closed, it is a triviality (using invariant theory) to show that all the symmetric powers of the natural $SL_n(k)$-module are indecomposable. Doty [12] has described all the submodules of these modules (see also [9]). Returning to the case where k is finite, Doty and Walker [13] have described the composition factors of the symmetric algebra of the natural $SL_3(\mathbb{F}_p)$-module.

There are a number of general results on the symmetric powers of the representation of a finite group. A classical example is Molien's theorem [20, 5], which can be used to calculate the irreducible components. The work of Howe [18], Bryant [8], and the second author [24] is more closely related to the results of this paper; their results show that the symmetric powers of a representation of a finite group are made up mostly of free modules in a certain technical sense.

1.3. *Overview of the Proofs*

In this paper we prove a generalization of the result of Glover, Alperin, and Kovacs from 2×2 to 3×3 matrices, and describe our plan for a generalization to $n \times n$ matrices for any n. This plan, if it can in fact be carried out, would prove the finiteness conjecture above. The results of this paper show that the finiteness conjecture is true for kG-modules of k-dimension 3 or less.

Recall that any kG-module M is a direct summand of the induced module $\mathrm{Ind}^G \mathrm{Res}_P M$, where P is a Sylow p-subgroup of G. Thus a result for p-groups only will still yield a lot of information about the general case, and, in particular, to prove the finiteness conjecture it is sufficient to obtain the result for p-groups.

On the other hand, any three-dimensional representation over k of a p-group must factor through the canonical representation of $U_3(k)$, the group of 3×3 upper triangular unitary matrices over k. (This is because $U_3(k)$ is the Sylow p-subgroup of $GL_3(k)$.) It follows that any decomposition theorem that we can prove for this representation of $U_3(k)$ will yield one for all p-groups upon restriction.

We believe, and hope to prove in a sequel, that there is a natural generalization of our results to any n. To indicate more clearly what sort of result we have in mind, we restate here, in the language of our main theorem, a special case of the result of Glover, Alperin, and Kovacs. Let G

be a p-group, let M be a two-dimensional kG-module, and let $S = k[x, y]$ be the symmetric algebra of M.

THEOREM 1.1. *There is an isomorphism of kG-modules*

$$S \approx \left(k\left[d_y\right] \otimes_k B\right) \oplus \left(k\left[d_x, d_y\right] \otimes_k P\right),$$

where $B = \bigoplus_{r=0}^{q-2} S^r$ *and* $P \subset S^{q-1}$ *is free of rank* 1.

It is clear that the generalization to dimension n should describe $k[x_1, \dots, x_n]$ as a kG-module by giving 2^{n-1} submodules, corresponding to the 2^{n-1} subsets of the set of polynomial invariants $\{d_1, \dots, d_{q^{n-1}}\}$ containing $d_{q^{n-1}}$. The submodule corresponding to a set of invariants will be propagated by those invariants and free over a certain subgroup.

The relative simplicity of the two-variable case (Theorem 1.1) can be seen as resulting from the fact that for 2×2 matrices there are only two "parts": a periodic part and a projective part, and both are quite easy to describe.

1.4. *Future Developments*

We hope to extend the results of this paper to $n \times n$ matrices for all n in a sequel. Since our type of result (for $U_n(k)$) guarantees a similar type of result for $GL_n(k)$, it would be interesting to obtain the precise form of the analog. Also, though it is clear that the result for a specific group $G \subset GL_n(k)$ can be obtained by restriction, there is some chance that sharper results can be obtained.

2. STUDYING THE UPPER TRIANGULAR GROUP

In this section we let G be the group of upper triangular unitary matrices over k, acting by its defining representation on $V \approx k^3$. Denote the vectors of the standard basis of V by x, y, and z, where we have chosen the names of these vectors so that G preserves the following flag of k-subspaces of V: $\langle x \rangle \subset \langle x, y \rangle \subset \langle x, y, z \rangle$. For the sake of clarity we note that with the chosen notation the socle of the kG-module V is $\langle x \rangle$. We denote by H the subgroup of G consisting of elements which fix z.

Let S be the symmetric algebra of V; S can be naturally identified with the polynomial algebra $k[x, y, z]$, and we note that $\operatorname{soc}(S) = k[x, y, z]^G = k[d_x, d_y, d_z]$, where $d_x = x$, $d_y = \prod_{\alpha \in k}(y - \alpha x) = y^q - x^{q-1}y$, $d_z = \prod_{v \in \langle x, y \rangle}(z - v)$.

Remark. The identification of the socle with the invariant ring follows from the fact that G is a p-group. The description of the invariant ring follows from a theorem of Nakijma and Stong, which can be found in [21, Theorem 5.5.6].

Our main theorem, stated in the Introduction, describes the structure of S as a kG-module. The proof of this theorem is somewhat involved and will occupy most of the next two sections. For the reader's convenience, we give a brief outline here.

First, we show that $k[d_z]$ can be "factored out," i.e., that $S \approx k[d_z] \otimes_k T$ for a certain $k[d_x, d_y][G]$-module T which we describe. Next we produce the free module P. Then, using a variation on a theorem of Feshbach [14] and also of Priddy and Wilkerson [19], we embed T in a doubly infinite sum of free kG-modules. Using this embedding, we define the submodules L and R of the main theorem. These submodules are also part of a decomposition of T, and the most difficult part of the proof is to obtain a decomposition of T using L and R. We show how all the parts of the proof fit together in Section 4.

2.1. *Factoring out $k[d_z]$*

We begin by noting that each kG-module S^r has a natural filtration by z-degree.

DEFINITION 2.1.

$$S^r[i] = \{f \in S^r \mid \deg_z(f) \le i\}, \qquad S^r(i) = \{f \in S^r \mid \deg_z(f) = i\}.$$

The reader should note that while the $S^r(i)$ are merely kH-modules the $S^r[i]$ are kG-submodules and form an increasing filtration of S^r. The next lemma is an equally simple observation.

LEMMA 2.2. *If $f \in S^G$ and M is a kG-submodule of S then fM is a kG-submodule of S.*

We now make a few more observations on this theme, which we formalize into lemmas for convenient reference in the proofs following.

Notation. We denote by M^r the set of degree r monomials in $k[x, y]$; for example, $M^2 = \{x^2, xy, y^2\}$.

Remark. With this notation it is easy to write down a basis of $S^r[q^2 - 1]$, namely, $M^r, zM^{r-1}, \dots, z^{q^2-1}M^{r-q^2+1}$. This very simple fact is the essence of the proof of Lemmas 2.6 and 2.7.

Since d_z has leading term z^{q^2} when regarded as a polynomial in z with coefficients in $k[x, y]$, the kG-submodule $d_z S^r \subset S^{r+q^2}$ has a natural k-basis of polynomials whose leading terms (when regarded as polynomials in z) are exactly those monomials in x, y, z with z-degree $\geq q^2$. This proves the following:

LEMMA 2.3. *$d_z S^r$ is a complement to $S^{r+q^2}[q^2 - 1]$ as a kG-module, that is,*

$$S^{r+q^2} = S^{r+q^2}\left[q^2 - 1\right] \oplus d_z S^r.$$

In view of this we will study a "truncation" of S;

DEFINITION 2.4. $T = \bigoplus_{r \geq 0} S^r[q^2 - 1]$.

We note that it follows immediately from the definition that T is a kG-submodule, closed under multiplication by x, or by y. Since $d_x = x$, and d_y involves only x and y, it follows that T is a $k[d_x, d_y][G]$-submodule of S. The reader should note also that the filtration of S yields one for T, so that we have modules $T^r[i]$ and $T^r(i)$ as in Definition 2.1.

The next lemma reduces the study of S to the study of T, and follows from repeated application of Lemma 2.3.

LEMMA 2.5. *The $k[d_x, d_y, d_z][G]$-module map*

$$k[d_z] \otimes_k T \to S,$$

induced from the inclusions $k[d_z] \subset S$, $T \subset S$, is an isomorphism.

Noting that the definition of T allows us to work with monomial bases for the T^r, we can obtain quite easily the following facts:

LEMMA 2.6. *If $r > (q^2 + q - 2)$, $d_x T^{r-1} + d_y T^{r-q} = T^r$.*

Proof. The monomials $y^r, zy^{r-1}, \ldots, z^{q^2-1} y^{r-q^2+1}$ project to a k-basis of the quotient $T^r / d_x T^{r-1}$. If $r > (q^2 + q - 2)$, all of these monomials are divisible by y^q, and therefore in the image of $\cdot d_y \colon T^{r-q} \to T^r \to T^r / d_x T^{r-1}$, since d_y is congruent to y^q modulo x. ∎

A minor variation on the argument above proves:

LEMMA 2.7. *If $r = (q^2 + q - 2)$, then $T^r / (d_x T^{r-1} + d_y T^{r-q})$ is one-dimensional, and generated by the image of $z^{q^2-1} y^{q-1}$.*

2.2. *The Existence of the Projective Module*

In this subsection we prove a version (Theorem 2.12) of the theorem of Feshbach and Priddy and Wilkerson mentioned earlier. (See also [6]). First we need some preliminary lemmas. The first is immediate from the definitions, so we omit its proof.

LEMMA 2.8. *Let L be a group and let $\varphi\colon M \to N$ be a map of kL-modules. Then*:

(i) *φ is injective if and only if* $\operatorname{soc}(\varphi)\colon \operatorname{soc}(M) \to \operatorname{soc}(N)$ *is injective*.

(ii) *If M is an injective kL-module then φ is an isomorphism if and only if* $\operatorname{soc}(\varphi)\colon \operatorname{soc}(M) \to \cos(N)$ *is an isomorphism.*

LEMMA 2.9. *Let L be a p-group, k a field of characteristic p, and M a kL-module. Suppose that* $\operatorname{Tr}_L x \neq 0$ *for some $x \in M$. Then the kL-submodule $\langle x \rangle \subset M$ is free.*

Proof. We have a natural map $\varphi\colon kL \twoheadrightarrow \langle x \rangle$ given by welding $1 \in kL$ to x. Since L is a p-group $\langle \operatorname{Tr}_L \rangle \subset kL$ is $\operatorname{soc}(kL)$. Thus, by hypothesis, φ is injective on $\operatorname{soc}(kL)$, and therefore injective on kL by Lemma 2.8, so it is an isomorphism. ∎

LEMMA 2.10. *For any finite field k and indeterminates $t, x_1, \dots, x_n$,*

$$\sum_{(\lambda_1,\dots,\lambda_n)\in k^n} \frac{1}{t+\lambda_1 x_1+\cdots+\lambda_n x_n} \prod_{(\lambda_1,\dots,\lambda_n)\in k^n} (t+\lambda_1 x_1+\cdots+\lambda_n x_n)$$
$$= \prod_{(\lambda_1,\dots,\lambda_n)\in k^n\setminus\vec{0}} (\lambda_1 x_1+\cdots+\lambda_n x_n).$$

This lemma really is part of the theory of Dickson invariants; see, e.g., [5, 26, 21] for an exposition.

Proof. We regard

$$\prod_{(\lambda_1,\dots,\lambda_n)\in k^n} (t+\lambda_1 x_1+\cdots+\lambda_n x_n)$$

as a polynomial $\Pi(t)$ with coefficients in $k[x_1,\dots,x_n]$. An easy computation shows that $\Pi(t)$ has the property that, for any $\alpha \in k$, $\Pi(\alpha t) = \alpha \Pi(t)$. It follows that the only powers of t appearing in Π are t^{q^i}. We may compute the t-coefficient c_0 directly, and obtain

$$c_0(x_1,\dots,x_n) = \prod_{(\lambda_1,\dots,\lambda_n)\in k^n\setminus\vec{0}} (\lambda_1 x_1+\cdots+\lambda_n x_n).$$

We have thus shown that

$$\prod_{(\lambda_1,\dots,\lambda_n)\in k^n} (t+\lambda_1 x_1+\cdots+\lambda_n x_n) = t^{q^n} + c_{n-1} t^{q^n - q^{n-1}} + \cdots + c_0 t.$$

Differentiating this equation with respect to t, one obtains the desired result. ∎

PROPOSITION 2.11.

$$\operatorname{Tr}_G \frac{d_y}{y}\frac{d_z}{z} = \left(d_x^2 d_y\right)^{q-1}.$$

Proof.

$$\begin{aligned}
\operatorname{Tr}_G \frac{d_y}{y}\frac{d_z}{z} &= \sum_{(\lambda,\mu,\nu)\in k^3} \frac{d_z}{z+\lambda y+\mu x}\,\frac{d_y}{y+\nu x} \\
&= \sum_{(\lambda,\mu)\in k^2} \frac{d_z}{z+\lambda y+\mu x} \sum_{\nu\in k} \frac{d_y}{y+\nu x} \\
&= \prod_{(\lambda,\mu)\in k^2-(0,0)} (\lambda y+\mu x) \prod_{\nu\in k^\times} \nu x \qquad \text{(by Lemma 2.10)} \\
&= \left(\prod_{\lambda\in k^\times}\lambda\right)(d_y d_x)^{q-1}\left(\prod_{\nu\in k^\times}\nu\right) d_x^{q-1} \\
&= \left(d_x^2 d_y\right)^{q-1}.
\end{aligned}$$

∎

From Lemma 2.9 we obtain

COROLLARY. *There is a projective* (*in fact*, *free of rank* 1) *kG-module* $P \subset T^{q^2+q-2}$ *with* $\operatorname{soc} P = \langle (d_x^2 d_y)^{q-1}\rangle$.

This enables us to prove a fundamental localization result. T is a graded $k[d_x, d_y]$-module, so we can form

$$T\left[d_x^{-1}, d_y^{-1}\right] = k\left[d_x^{-1}, d_y^{-1}, d_x, d_y\right] \otimes_{k[d_x, d_y]} T.$$

This is a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ring $k[d_x^{-1}, d_y^{-1}, d_x, d_y]$ and is also a G-module.

THEOREM 2.12 (Feshbach, Priddy and Wilkerson, and Benson). *The natural map*

$$\varphi\colon k\left[d_x^{-1}, d_y^{-1}, d_x, d_y\right] \otimes P \to T\left[d_x^{-1}, d_y^{-1}\right]$$

is an isomorphism of $k[d_x^{-1}, d_y^{-1}, d_x, d_y][G]$-*modules*.

Remark. For various other applications of this theorem, see the references cited.

Remark. $T[d_x^{-1}, d_y^{-1}]$ has a finite filtration by z-degree, and so does P. The isomorphism of the theorem respects these filtrations.

Proof. $\operatorname{soc} k[d_x^{-1}, d_y^{-1}, d_x, d_y] \otimes P = k[d_x^{-1}, d_y^{-1}, d_x, d_y] \otimes \operatorname{soc} P$ is mapped isomorphically to $k[d_x^{-1}, d_y^{-1}, d_x, d_y](d_x^2 d_y)^{q-1}$ by Lemma 2.2

But $k[d_x^{-1}, d_y^{-1}, d_x, d_y](d_x^2 d_y)^{q-1} = k[d_x^{-1}, d_y^{-1}, d_x, d_y]$ is $\operatorname{soc} T[d_x^{-1}, d_y^{-1}]$, and so Lemma 2.8 completes the proof. ▮

DEFINITION 2.13. $Q \subset T$ is the $k[d_x, d_y][G]$-module generated by P, i.e., $Q = \bigoplus_{r \geq 0} Q^r$, where

$$Q^r = \bigoplus_{\substack{i,\, j \geq 0 \\ i + qj + q^2 + q - 2 = r}} d_x^i d_y^j P.$$

LEMMA 2.14. *For $r > q^2 - 3$, $\dim T^r - \dim Q^r = \dim T^{r'}$ where $r' \cong r \bmod q$ and $q^2 - 3 < r' \leq q^2 + q - 3$.*

Proof. Note that Q^r contains $(r - r')/q$ copies of P, so $\dim Q^r = (r - r')q^2$. On the other hand $T^r = \bigoplus_{i=0}^{q^2-1} T^r(i)$ so

$$\begin{aligned} \dim T^r &= \sum_{i=0}^{q^2-1} (r + 1 - i) = \sum (r - r') + \sum (r' + 1 - i) \\ &= q^2(r - r') + \dim T^{r'}. \end{aligned}$$

▮

We have now assembled enough facts to give a fairly simple proof of the "finiteness" part of the corollary mentioned in the Introduction without using our main theorem:

Proof. (Finitely Many Isomorphism Types of Indecomposable Modules). Write $T^r = Q^r \oplus X^r$. Then $\dim X^r < N$ for some N which does not depend on r. Since k is finite, there are only a finite number of isomorphism classes of kG-modules of dimension less than N. ▮

The next proposition is not necessary to our argument but is of independent interest.

PROPOSITION 2.15. *Q is a largest projective summand of T. In particular, there is no projective summand of T in degree less than $q^2 + q - 2$.*

Proof. $(d_y/y)(d_z/z)$ is not contained in $T[q^2 - 2]$, so $P \cap T[q^2 - 2]$ is a proper submodule of P and hence Tr_G vanishes on it. By the remark after Theorem 2.12 this implies that Tr_G vanishes on $T[q^2 - 2]$.

Thus any projective R in T^r must have nonzero image in $\overline{R} \subset T(q^2 - 1)$, regarded here as the quotient $T[q^2 - 1]/T[q^2 - 2]$. Since the projection

$T[q^2 - 1] \twoheadrightarrow T(q^2 - 1)$ splits as a map of kH-modules, R must be a projective kH-module of rank at least the rank of R as a kG-module. Now suppose that we have $Q^r \oplus P' \subset T^r$ for some projective P'. Then $\overline{Q}^r \oplus \overline{P}' \subset T^r(q^2 - 1)$ and counting dimensions we find that $r + r' + q \leq r + 2 - q^2$, or $r' \geq q^2 + q - 2$, which is a contradiction. ∎

Remark. This argument can easily be extended to the case of a polynomial ring in n variables to show that the lowest possible degree of an element of nonzero trace under $GL_n(k)$ is $1 + q + \cdots + q^{n-1} - n$. Since an n-variable analog of Proposition 2.11 shows that an element of nonzero trace actually occurs in this degree, we have answered a question of Wilkerson [25]. This question was answered previously in [10], as we discovered after proving the main result of this paper.

Remark. It is easy to see using Galois theory that T must contain a projective in some degree [2, 24]. This is all that is needed for our simple proof of the corollary mentioned in the Introduction.

3. DETAILS IN THE PROOF OF THE DECOMPOSITION

The purpose of this section is define the modules L and R and prove that they have the properties claimed in the statement of our main theorem. The statements of results in this section differ very slightly from the statement of the main theorem: here we define modules L^r and R^r in all degrees $\geq q^2 - 2$, and prove that the higher degree L^r's and R^r's can be obtained from ones in lower degree by multiplication by d_x and d_y, respectively.

Once this linguistic technicality is understood, we can state the following proposition, whose proof is the goal of this section.

PROPOSITION 3.1. *If* $r \geq q^2 - 2$, $T^r = L^r \oplus Q^r \oplus R^r$.

This proposition is the most difficult and essentially the only remaining point in the proof of the main theorem. Indeed, by combining it with the results of Section 2, we will derive the main theorem in Section 4.

3.1. *Defining L and R*

From Theorem 2.12 we have $T \to C = \bigoplus_{i,j \in \mathbb{Z}} d_x^i d_y^j P$. We think of this embedding as giving "coordinates" on T. More precisely, each $u \in C$ has a unique decomposition $u = \sum_{i,j} d_x^i d_y^j \theta_{ij}(u)$, where the θ_{ij} are "coordinate functions" in P.

The embedding also allows us to define the following submodules of C

$$Y_- = \bigoplus_{\substack{i \in \mathbb{Z} \\ j < 0}} d_x^i d_y^j P,$$

i.e., the $k[d_x, d_x^{-1}, d_y^{-1}][G]$-submodule generated by $d_y^{-1}P$, and

$$X_- = \bigoplus_{\substack{i < 0 \\ j \in \mathbb{Z}}} d_x^i d_y^j P,$$

i.e., the $k[d_x^{-1}, d_y, d_y^{-1}][G]$-submodule generated by $d_x^{-1}P$. Clearly X_- and Y_- are the direct sums of their homogeneous parts.

We now have the following

DEFINITION 3.2 (L and R). For $r \geq q^2 - 2$, let $L^r = Y_-^r \cap T^r$, $R^r = X_-^r \cap T^r$.

We now show that multiplication by d_x and d_y allows us to obtain all of L and R from the parts in small degrees. To do this we need the following preliminary lemma from commutative algebra.

LEMMA 3.3. *Let A be a UFD and let a, b be elements of A with no common factor. Let $f \in A$ and suppose that $f = ag$ for some $g \in A[b^{-1}]$. Then $g \in A$.*

Proof. We have $g = b^{-n}\tilde{g}$ for some $\tilde{g} \in A$, and we may assume that $b \nmid \tilde{g}$ in A. Therefore $b^n f = a\tilde{g}$ is an equation in A. Since b cannot divide the right-hand side, $n = 0$ and $\tilde{g} = g$, so $g \in A$. ∎

LEMMA 3.4. *If $r \geq q^2 - 2$, then $d_x L^r = L^{r+1}$ and $d_y R^r = R^{r+q}$.*

Proof. Clearly $d_x L^r \subset L^{r+1}$ and $d_y R^r \subset R^{r+q}$. We need to prove equality. We prove $d_x L^r = L^{r+1}$; the proof for R is much the same. Let $f \in L^{r+1}$ and $r \geq q^2 - 2$. Then we have $f = \sum d_x^i d_y^j p_{ij}$, where $i + qj + q^2 + q - 2 = r + 1$. Since $f \in L$, this sum is over $j < 0$. Since $r \geq q^2 - 2$, we have $i + q(j + 1) + q^2 - 2 \geq q^2 - 2 + 1$. Simplifying and noting that $q(j + 1) \leq 0$, we obtain $i \geq 1$. Thus $f = d_x g$ for some $g \in Y_- \cap T[d_y^{-1}]$. From the explicit expressions for d_x and d_y given at the beginning of this section, it is clear that d_x and d_y have no common factor in $k[x, y, z]$. Therefore, by lemma 3.3, we must have $g \in T$, and so $g \in L^r$. This completes the proof. ∎

We have now defined submodules L^r, R^r, and Q^r of T^r, and shown that they have the properties claimed in the first paragraph of this section. To prove Proposition 3.1, we must show that the natural map $L^r \oplus Q^r \oplus R^r$

$\to T^r$ is an isomorphism. Since it is an easy consequence of the definitions, we show first:

LEMMA 3.5. *If $r \geq q^2 - 2$, the natural map $L^r \oplus Q^r \oplus R^r \to T^r$ is injective.*

Proof. Note that

$$C^r = \bigoplus_{i+qj+q^2+q-2=r} d_x^i d_y^j P$$

is a direct sum $X_-^r \oplus Q^r \oplus Y_-^r$ if $r \geq q^2 - 2$, since modules $d_x^i d_y^j P$ for *both* i and j negative have degree less than $q^2 - 2$. This direct sum decomposition gives us an injection $(T \cap X_-^r) \oplus (T \cap Q^r) \oplus (T \cap Y_-^r) \to (T \cap C^r)$. But this injection is just the natural map $L^r \oplus Q^r \oplus R^r \to T^r$. ∎

Thus, we have seen that the linear independence part of Proposition 3.1 is an easy consequence of the definitions of L, R, and Q. To show that T is actually the sum is more difficult, however. Indeed, at this point it is not completely clear that L and R are nonzero, and the rest of the proof of Proposition 3.1 will be considerably more involved. Our approach will be to define vector space complements $\tilde{L}$ and $\tilde{R}$; instead of showing that L and R are "large enough" we show that $\tilde{L}$ and $\tilde{R}$ are "small enough."

DEFINITION 3.6 ($\tilde{L}$ and $\tilde{R}$). Let $\tilde{L}^{q^2-2}$ be a k-vector space complement for R^{q^2-2} in T^{q^2-2} which contains L^{q^2-2}, i.e.,

$$\tilde{L}^{q^2-2} \oplus R^{q^2-2} = T^{q^2-2}$$

as k-vector spaces and $\tilde{L}^{q^2-2} \supset L^{q^2-2}$.

For $(q^2 - 2) \leq r < (q^2 + q - 2)$ let $\tilde{R}^r$ be a k-vector space complement for L^r in T^r which contains R^r, i.e.,

$$\tilde{R}^r \oplus L^r = T^r$$

as k-vector spaces and $\tilde{R}^r \supset R^r$.

We extend the definition by setting $\tilde{L}^r = d_x \tilde{L}^{r-1}$ for $r > q^2 - 2$, and $\tilde{R}^r = d_y \tilde{R}^{r-q}$ for $r \geq q^2 + q - 2$. Finally, let $\tilde{R}^r = \tilde{L}^r = 0$ for $r < q^2 - 2$.

Notice that with the definitions just given $\tilde{L}$ and $\tilde{R}$ are homogeneous submodules of $T[d_x^{-1}, d_y^{-1}]$, closed under multiplication by d_x and d_y, respectively. We now prove that $\tilde{L}$ and $\tilde{R}$ do not meet Q.

LEMMA 3.7. *$\tilde{L} \cap Q = 0$ and $\tilde{R} \cap Q = 0$.*

Proof. We prove the property for $\tilde{L}$; the proof for $\tilde{R}$ is similar and left to the reader. Let $f \in \tilde{L} \cap Q$; we must show that $f = 0$. Since $\tilde{L}$ and Q

are homogeneous, we may assume that f is also. Therefore we may take $f = d_x^r g$ for some $g \in \tilde{L}^{q^2-2}$. Because $f \in Q$, we obtain $g = d_x^{-r}\varphi$ for some $\varphi \in Q$. But $d_x^{-r}Q \subset X_-$, and $g \in T$, so we have $g \in R$ by the definition of R. Now we have $g \in R \cap Q$ and so $g = 0$, which implies that $f = 0$. ∎

3.2. *The Proof of Proposition* 3.1

In this section we will prove that $\tilde{L}^r$ and $\tilde{R}^r$ are "small enough," after which the rest of the proof of Proposition 3.1 is a very easy dimension-counting argument. We will first show that $T^r = \tilde{L}^r + Q^r + \tilde{R}^r$, and then show that there is no linear relation between elements of $\tilde{L}^r$, Q^r, and $\tilde{R}^r$. This will actually prove that $T^r = \tilde{L}^r \oplus Q^r \oplus \tilde{R}^r$, after which the proof that $T^r = L^r \oplus Q^r \oplus R^r$ (Proposition 3.1) will be easy.

LEMMA 3.8. *If $r = (q^2 + q - 2)$, then $d_x R^{r-1} \subset R^r + P$ and $d_y L^{r-q} \subset L^r + P$.*

Proof. We prove the property only for R as the proof for L is very similar. Recall that P is homogeneous of degree $q^2 + q - 2$. From this it follows that

$$R^{r-1} = \left(d_x^{-1}P \oplus d_x^{-1-q}d_y P \oplus \cdots\right) \cap T^{r-1},$$

$$R^r = \left(d_x^{-q}d_y P \oplus d_x^{-2q}d_y^2 P \oplus \cdots\right) \cap T^r.$$

We can then observe that $d_x R^{r-1} \subset (P \oplus d_x^{-q}d_y P \oplus d_x^{-2q}d_y^2 P \oplus \cdots) \cap T^r = P + R^r$. (In the final equality, we have used the fact that $(P \oplus X_-) \cap T^r = P \oplus (X_- \cap T^r)$ since $P \subset T$.) ∎

Now we can prove that $T^r = \tilde{L}^r + Q^r + \tilde{R}^r$.

LEMMA 3.9. $T^r = \tilde{L}^r + Q^r + \tilde{R}^r$.

Proof. *Case* 1. Suppose $(q^2 - 2) \le r < (q^2 + q - 2)$. Then $Q^r = 0$, and we have to show that $\tilde{L}^r + \tilde{R}^r = T^r$. From Definition 3.6, we have that $\tilde{L}^r \supset L^r$, so $\tilde{L}^r + \tilde{R}^r \supset L^r + \tilde{R}^r$, which is T^r since R^r is by definition a complement to L^r in this range.

Case 2. Suppose $n = q^2 + q - 2$. Then we have to show that $\tilde{L}^r + P + \tilde{R}^r = T^r$. From our observations on monomial bases (Lemma 2.7) we obtain $d_x T^{r-1} + d_y T^{r-q} + P = T^r$. Then, noting that $T^{r-1} = \tilde{L}^{r-1} \oplus R^{r-1}$, we have $d_x T^{r-1} = d_x \tilde{L}^{r-1} + d_x R^{r-1}$. Similarly, $d_y T^{r-q} = d_y L^{r-q} + d_y \tilde{R}^{r-q}$.

Thus we have

$$\begin{aligned}
T^r &= d_x T^{r-1} + d_y T^{r-q} + P \\
&= d_x \tilde{L}^{r-1} + d_x R^{r-1} + d_y \tilde{R}^{r-q} + d_y L^{r-q} + P \\
&= \tilde{L}^r + \tilde{R}^r + d_x R^{r-1} + d_y L^{r-q} + P \\
&= \tilde{L}^r + \tilde{R}^r + R^r + L^r + P \\
&= \tilde{R}^r + \tilde{L}^r + P.
\end{aligned}$$

Case 3. Suppose $r > q^2 + q - 2$. Then we proceed by the method of Case 2, but using Lemma 2.6 instead of Lemma 2.7. ∎

Now let X be a finite dimensional k-vector space contained in C. Recall that a vector $u \in X$ has "coordinates" $\theta_{ij}(u) \in P$. If $X \subset C^r$, these coordinates are nonzero only for those i, j satisfying $i + qj + q^2 + q - 2 = r$; i.e., each $u \in X \subset C^r$ has a unique decomposition $u = \sum_i d_x^i d_y^{j(i)} \theta_i(u)$ where $\theta_i(u) \in P$. Given a particular $u \in X$, we call the collection $\{i \in \mathbb{Z} | \theta_i(u) \neq 0\}$ the *support* of u, or $\operatorname{supp} u$.

DEFINITION 3.10. The *support* of X is the union $\bigcup_{u \in X} \operatorname{supp}(u)$. The *width* of X is $\max(\operatorname{supp} X) - \min(\operatorname{supp} X)$.

It is easy to show that if X and Y are two finite-dimensional vector spaces, we have $\operatorname{supp}(X + Y) = \operatorname{supp}(X) \cup \operatorname{supp}(Y)$ and $\operatorname{supp}(X \cap Y) \subset \operatorname{supp}(X) \cap \operatorname{supp}(Y)$. Note also that $d_x X$ and $d_y X$ are also finite-dimensional k-vector spaces, and that $\operatorname{supp}(d_x X) = \operatorname{supp}(X) + 1$ and $\operatorname{supp}(d_y X) = \operatorname{supp}(X)$. It follows that $\operatorname{width}(d_x X) = \operatorname{width}(X)$, $\operatorname{width}(d_y X) = \operatorname{width}(X)$.

Thus, from the definition of $\tilde{L}$ and $\tilde{R}$, it is clear that we may choose N so large that $\operatorname{supp} \tilde{L}^N \cap \operatorname{supp} \tilde{R}^N = \emptyset$. For this N we will have

$$\operatorname{supp}(\tilde{L}^N) \subset [-m, m], \qquad \operatorname{supp}(\tilde{R}^N) \subset [M - m, M + m],$$
$$\operatorname{supp}(Q^N) = [0, M]$$

for some $m < \frac{1}{2}M$. Here, $M = N - (q^2 + q - 2)$.

Recall that the definition of Q tells us that Q is a direct sum of translates of the free module P. We note the following consequence of this fact:

LEMMA 3.11. *Suppose* $f \in T^N$ *and* $\operatorname{supp} f \subset [0, M]$. *Then* $f \in Q^N$.

We can now show that there is no linear relation among $\tilde{L}$, Q, and $\tilde{R}$.

LEMMA 3.12. *Suppose that* $\alpha + \lambda + \rho = 0$ *is a linear relation with* $\alpha \in Q^N$, $\lambda \in \tilde{L}^N$, *and* $\rho \in \tilde{R}^N$. *Then* $\alpha = \lambda = \rho = 0$.

Proof. Rewriting our relation as $\lambda = -\alpha - \rho$, we see that

$$\operatorname{supp} \lambda \subset [0, M] \cup [M-m, M+m].$$

Since $\lambda \in \tilde{L}^N$, we have

$$\operatorname{supp} \lambda \subset ([0, M] \cup [M-m, M+m]) \cap [-m, m] = [0, m].$$

So, by Lemma 3.11, we have $\lambda \in Q^N$, so $\lambda \in Q \cap \tilde{L}$, which is zero by Lemma 3.7. It follows that $\rho \in Q \cap \tilde{R}$, which is also zero. Therefore $\alpha = 0$ as well. ∎

We observe now that from Lemmas 3.9 and 3.12 it follows that $S^N = \tilde{L}^N \oplus Q^N \oplus \tilde{R}^N$ for large N, so that $\dim T^N = \dim \tilde{L}^N + \dim Q^N + \dim \tilde{R}^N$ for large N. From this we can show that the same dimension formula holds for all $r \geq q^2 - 2$:

LEMMA 3.13. *If $r \geq q^2 - 2$, $\dim T^r = \dim \tilde{L}^r + \dim Q^r + \dim \tilde{R}^r$.*

Proof. From the argument in Lemma 2.14 it is easy to see that $\dim T^{r+q} = \dim T^r + q^3$ and that $\dim Q^{r+q} = \dim Q^r + q^3$ for $r \geq q^2 - 2$. Furthermore, we have $\dim \tilde{L}^{r+q} = \dim \tilde{L}^r$ and $\dim \tilde{R}^{r+q} = \dim \tilde{R}^r$ for $r \geq q^2 - 2$ from Definition 3.6.

Adding these equations, we see that if $\dim T^{r+q} = \dim \tilde{L}^{r+q} + \dim Q^{r+q} + \dim \tilde{R}^{r+q}$, for some $r \geq q^2 - 2$, then $\dim T^r = \dim \tilde{L}^r + \dim Q^r + \dim \tilde{R}^r$.

Since Lemma 3.12 shows that the desired dimension formula is true for all sufficiently large r, the proposition follows by descent. ∎

From Lemmas 3.9 and 3.13 it follows that $T^r = \tilde{L}^r \oplus Q^r \oplus \tilde{R}^r$ for $r \geq q^2 - 2$. We can now count dimensions to prove:

LEMMA 3.14. *If $r \geq q^2 - 2$, $\dim T^r = \dim L^r + \dim Q^r + \dim R^r$.*

Proof. From Definition 3.6, we have $\dim \tilde{L}^r + \dim R^r = \dim T^r$ and $\dim \tilde{R}^r + \dim L^r = \dim T^r$ if $q^2 - 2 \leq r < q^2 + q - 2$. Thus, for r in this range, we have $\dim \tilde{L}^r + \dim \tilde{R}^r = 2 \dim T^r - \dim R^r - \dim L^r$. It follows by the dimension formula of Lemma 3.13 that $\dim T^r = \dim R^r + \dim L^r$, so the lemma is proved for r in this range. It follows from the inductive dimension formulas developed in Lemma 3.13 that $\dim T^r = \dim L^r + \dim Q^r + \dim R^r$ for all $r \geq q^2 - 2$. ∎

Proposition 3.1 is now an immediate consequence of Lemmas 3.14 and 3.5.

4. THE THEOREM

We now have the (very easy) task of assembling all the work we have done to obtain our main theorem.

Proposition 3.1 tells us that $T^r = L^r \oplus Q^r \oplus R^r$ in degree $r \geq q^2 - 2$. By Lemma 3.4,

$$\bigoplus_{r \geq q^2-2} L^r = k[d_x] \otimes_k L^{q^2-2} \qquad \text{and}$$

$$\bigoplus_{r \geq q^2-2} R^r = k\left[d_y\right] \otimes_k \left(\bigoplus_{r=q^2-2}^{q^2+q-3} R^r \right),$$

so if we define

$$B = \bigoplus_{r=0}^{q^2-3} T^r \left(= \bigoplus_{r=0}^{q^2-3} S^r \right), \qquad L = L^{q^2-2}, \qquad R = \left(\bigoplus_{r=q^2-2}^{q^2+q-3} R^r \right),$$

then using the definition of Q we have

$$T = B \oplus \left(k[d_x] \otimes_k L\right) \oplus \left(k\left[d_y\right] \otimes_k R\right) \oplus \left(k\left[d_x, d_y\right] \otimes_k P\right).$$

We now use the fact that $S = k[d_z] \otimes_k T$ (Lemma 2.5). This proves:

THEOREM 4.1. *There is an isomorphism of kG-modules*

$$\begin{aligned} S \approx &\left(k[d_z] \otimes_k B\right) \oplus \left(k[d_x, d_z] \otimes_k L\right) \\ &\in \left(k\left[d_y, d_z\right] \otimes_k R\right) \oplus \left(k\left[d_x, d_y, d_z\right] \otimes_k P\right). \end{aligned}$$

COROLLARY 4.2. *For any group H (not necessarily a p-group) and any kH-module M with $\dim_k M \leq 3$, the symmetric algebra of M contains only a finite number of isomorphism types of indecomposable summands as a kH-module.*

Proof. This follows immediately from the theorem above and the remarks in Section 1.3. ∎

COROLLARY 4.3. *For any r, there is an isomorphism of kG-modules*

$$S^r \approx P^{\alpha(r)} \oplus \bigoplus_{i=0}^{q^2+q-3} \left(S^i\right)^{\beta(i,r)},$$

where the multiplicities α, β are given most succinctly in terms of Poincaré series:

$$\sum_{r=0}^{\infty} \alpha(r)t^r = \frac{t^{q^2+q-2}}{(1-t)(1-t^q)(1-t^{q^2})}$$

and

$$\sum_{r=0}^{\infty} \beta(i,r)t^r = \begin{cases} \dfrac{t^i}{(1-t^{q^2})}, & i \le q^2-3, \\ \dfrac{t^i}{(1-t^q)(1-t^{q^2})}, & q^2-2 \le i \le q^2+q-3. \end{cases}$$

Furthermore, the same is true for any p-group Q and any three-dimensional kQ-module.

Proof. The result for general p-groups follows from the result for G by restriction. To prove the corollary for G, note that the S^i for $i \le q^2 - 3$ give the B part in Theorem 4.1. The S^i for $q^2 - 2 \le i \le q^2 + q - 3$ give the L and R part because in this range $S^i = L^i \oplus R^i$. ∎

Remark. Let H_2 be the subgroup of G consisting of all automorphisms of S of the form $z \mapsto z + \lambda y$, where $\lambda \in k$. Let H_1 be the subgroup of G consisting of all automorphisms of the form $z \mapsto z + \lambda x$, $y \mapsto y + \mu x$, where $\lambda, \mu \in k$. If k^+ denotes the additive group of k, we have $H_2 \approx k^+$ and $H_1 \approx k^+ \times k^+$. It can be shown that L is induced from H_2 and R is induced from H_1. The proof of this fact is not important to our goals in this paper and we have therefore chosen to defer it to a future publication.

We conclude our treatment of this theorem with a short table, in which we indicate the $U_3(k)$-summands of $k[x, y, z]$ in small degrees for the field $k = \mathbb{F}_2$. We denote modules by B, L, R, or P to indicate the part of the decomposition in which they arise; the subscripts give the k-dimensions.

Degree	Summands			
0	k			
1	B_3			
2	L_4	R_2		
3	L_4	R_6		
4	L_4	R_2	P_8	k
5	L_4	R_6	P_8	B_3

It is of course possible to extend this table as far as desired using the theorem.

5. COMMENTARY ON DIFFICULTIES OF THE PROOF

The main theorem of this paper is a direct sum decomposition of the polynomial ring $k[x, y, z]$ as a G-module, for which we have given a somewhat convoluted proof. The reader may wonder whether all of this effort is necessary, especially given the relative simplicity of the result for $n = 2$ and the decomposition (Theorem 2.12) $T[d_x^{-1}, d_y^{-1}] = \oplus d_x^i d_y^j P$.

In particular one might naively wonder whether the decomposition we give is any different from attempting to write $T = \oplus(d_x^i d_y^j P \cap T)$. It turns out that this rather optimistic attempt to prove the theorem works for $q = 2$, and we indicate here why this method fails for general q.

Suppose $q = 5$. Then there is a decomposition $S^{20} = T^{20} = P_5 \oplus M_{201} \oplus P_{25}$, where P_5 and P_{25} are permutation modules and M_{201} is an indecomposable module of k-dimension 201. If the naive attempt at a proof indicated above worked for $q = 5$, every summand of T would have k-dimension bounded by the k-dimension of P, which is q^3, or 125 in this case.

As a computationally simpler perspective on the failure of this attempt at a proof, we can consider the case $q = 3$. In this case P is homogeneous of degree 10, and has socle $\langle d_y^2 d_x^4 \rangle$. Thus we have $(T^3 \cap d_x^{-4} d_y^{-1} P) \supset \langle d_y \rangle \neq 0$ and $(T^3 \cap d_x^{-1} d_y^{-2} P) \supset \langle d_x^3 \rangle \neq 0$, so, if the hypothetical decomposition held, $T^3 = S^3$ would be a nontrivial direct sum. In fact, however, T^3 is indecomposable.

In spite of these facts, we believe that the idea of attempting to write $T = \oplus(d_x^i d_y^j P \cap T)$ has conceptual value. In particular, for $q = 2$ it corresponds to a "factorization"

$$k[x, y, z] \approx k[d_z] \otimes Y \otimes X,$$

where Y is a $k[d_y][G]$-module with Poincaré series $1 + 2t^2 + 2t^4 + 2t^6 + \cdots$ and X is a $k[d_x][G]$-module with Poincaré series $1 + 3t + 4t^2 + 4t^3 + \cdots$. This type of factorization is not evident for $q \neq 2$, but the search for such a factorization was the avenue that led to the theorem we have presented in this paper. Furthermore, a partial factorization based on this can be used to prove our theorem for $n = 4$ and $q = 2$.

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